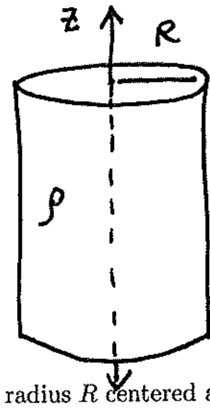
PHYS 2020A: Practice Midterm 1

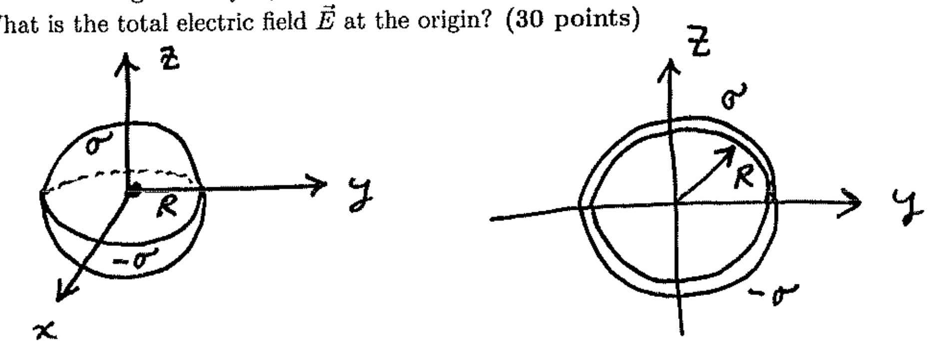
Problem 1: Consider an infinite cylinder of radius R and uniform charge density ρ aligned along the z axis.

(a) What is the electric field $\vec{E}$ at a distance s from the z axis? **(20 points)**

(b) How much work does it take to move a point charge q from a point on the outside surface of the cylinder to a point in the center? **(15 points)**



Problem 2: Consider a spherical shell of radius R centered at the origin. Suppose that the top hemisphere ($z > 0$) has uniform surface charge density σ, while the bottom hemisphere ($z < 0$) has uniform surface charge density $-\sigma$. What is the total electric field $\vec{E}$ at the origin? **(30 points)**

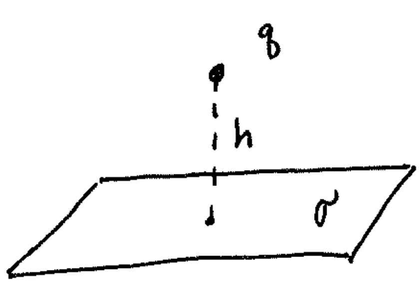


Problem 3: Consider a point charge q fixed at height h above an infinite plane with surface density σ.

(a) Suppose we introduce an *additional* point charge q. At what position would this second charge q be in equilibrium? Assume that q and σ are both positive. **(15 points)**

(b) Is the equilibrium position stable with respect to displacements parallel to the plane? Justify your answer. **(10 points)**

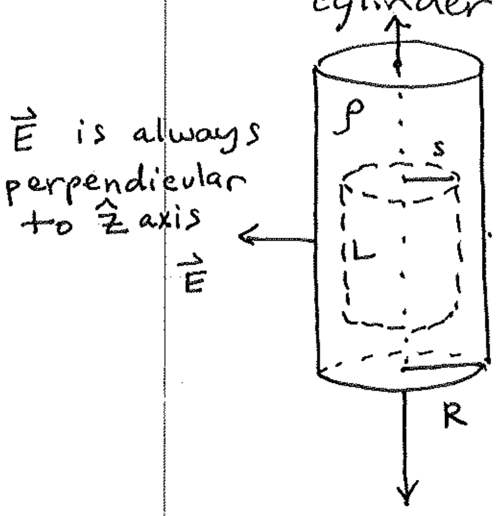
(c) Is the equilibrium position stable with respect to displacements perpendicular to the plane? Justify your answer. **(10 points)**



Practice Midterm solutions

Problem 1

(a) Draw Gaussian surface ~~of~~ ~~ea~~ as a cylinder of radius s and height L.



case $s<R$: Gauss's Law

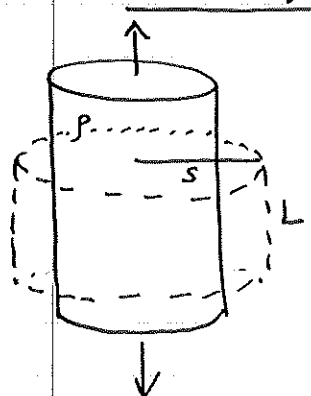
$$\int d\vec{A}\cdot\vec{E} = EA = \frac{Q_{encl}}{\varepsilon_0} = \frac{\rho V}{\varepsilon_0}$$

$A = 2\pi s L$ = side area of Gaussian surface

$V = \pi s^2 L$ = volume of Gaussian surface

$$\Rightarrow \quad E(s) = \frac{\rho V}{A\varepsilon_0} = \frac{\rho \pi s^2 L}{2\pi s L \varepsilon_0} = \frac{\rho s}{2\varepsilon_0}$$

case $s>R$: same but now $V = \pi R^2 L$



$$E(s) = \frac{\rho V}{A\varepsilon_0} = \frac{\rho R^2}{2s\varepsilon_0}$$

Putting in the direction:

$$\boxed{\vec{E} = \begin{cases} \dfrac{\rho s}{2\varepsilon_0}\hat{s} & s<R \\[2ex] \dfrac{\rho R^2}{2s\varepsilon_0}\hat{s} & s>R \end{cases}}$$

(b) Work from $s=R$ to $s=0$

$$W = -\int_R^0 \vec{ds}\cdot\vec{F} \qquad \vec{ds} = ds\,\hat{s},\ \vec{F} = q\vec{E} = \frac{\rho s\,\hat{s}}{2\varepsilon_0}q$$

since $s \le R$.

So the work is

$$W = -\int_R^0 ds\ \frac{q\rho s}{2\varepsilon_0} = \underline{\frac{q\rho R^2}{4\varepsilon_0}}$$

Note: positive work required if ρ & q have the same sign.

<u>Problem 2</u>:

Use $\displaystyle \vec{E} = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$

Here $dq = \begin{cases} \sigma\, dA & \text{upper half} \\ -\sigma\, dA & \text{lower half} \end{cases}$

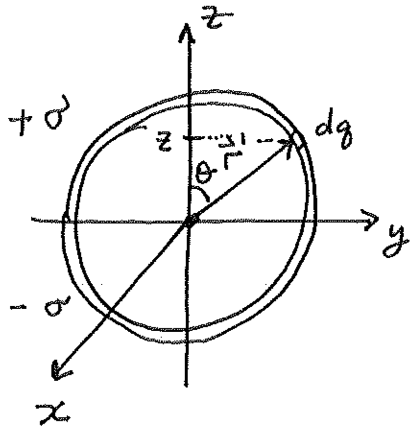
In spherical coords, $dA = R^2 \sin\theta\, d\theta\, d\phi$

Check: Surface area of sphere is

$$A = \int dA = \int R^2 \sin\theta\, d\theta\, d\phi$$

$$= R^2\int_0^{\pi} d\theta\,\sin\theta \int_0^{2\pi} d\phi = 4\pi R^2$$

By symmetry, $\vec{E}$ points in $\hat{z}$ direction.



Position vectors:

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = -\vec{r}'$ since

we want $\vec{E}$ at $\vec{r} = (0,0,0)$.

Note $|\vec{r}'| = R$

Then $\mathscr{r} = |\vec{\mathscr{r}}| = R$ also.

The z-component of $\hat{\mathscr{r}}$ is $-z/R = -\cos\theta$

$$E_z = \int \frac{1}{4\pi\varepsilon_0} \frac{\sigma\, dA}{R^2} (-\cos\theta) \quad \leftarrow \text{upper half only}$$

$$= -\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \, \sin\theta \cos\theta \, \frac{\sigma}{4\pi\varepsilon_0} = -\frac{\sigma}{4\varepsilon_0}$$

$$E_z = \int \frac{1}{4\pi\varepsilon_0} \frac{(-\sigma)\, dA}{R^2} (-\cos\theta) \quad \leftarrow \text{lower half only}$$

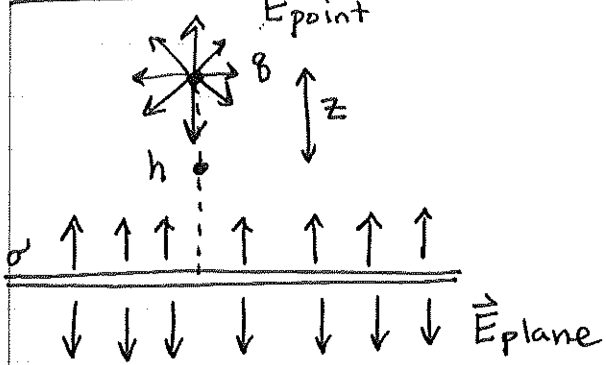
$$= +\int_{\pi/2}^{\pi} d\theta \int_0^{2\pi} d\phi \, \sin\theta \cos\theta \, \frac{\sigma}{4\pi\varepsilon_0} = -\frac{\sigma}{4\varepsilon_0}$$

Adding them together: $E_z = -\dfrac{\sigma}{2\varepsilon_0}$

$$\boxed{\vec{E} = -\frac{\sigma}{2\varepsilon_0}\hat{z}}$$

Problem 3

(a)



Now introduce another point charge q.
Only place where total force

$$\vec{F} = q\vec{E}_{point} + q\vec{E}_{plane}$$

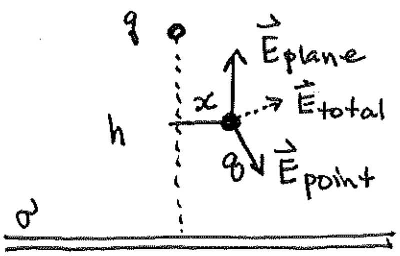
vanishes is along the line h.

Let z be distance along h from point charge q.

$$\vec{F} = 0 = q\left(-\frac{q}{4\pi\varepsilon_0 z^2}\hat{z}\right) + q\left(\frac{\sigma}{2\varepsilon_0}\hat{z}\right)$$

Vanishes when $\boxed{z = \sqrt{\frac{q}{2\pi\sigma}}}$ ← equilibrium point

(b)

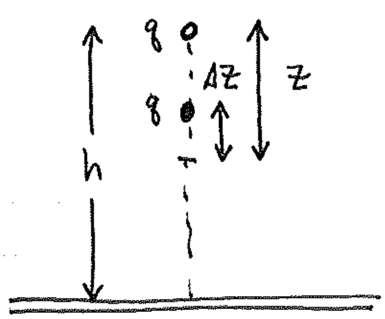


Displace 2nd charge q by distance x.

The net force on q is $\vec{F} = q\vec{E}_{total}$ points away from the equilibrium point in part (a).

⇒ Unstable

(c)



Consider displacing 2nd charge q vertically by Δz from equilibrium point at z.

$$\vec{F} = q\left(-\frac{q}{4\pi\varepsilon_0(z-\Delta z)^2} + \frac{\sigma}{2\varepsilon_0}\right)\hat{z}$$

If $\Delta z > 0$ (q moves up by Δz), then

$$\left(-\frac{q}{4\pi\varepsilon_0(z-\Delta z)^2} + \frac{\sigma}{2\varepsilon_0}\right) < 0$$

So F_z is negative & $\vec{F}$ points down.

If $\Delta z < 0$ (q moves down by $|\Delta z|$), then

$$\left(-\frac{q}{4\pi\varepsilon_0(z-\Delta z)^2} + \frac{\sigma}{2\varepsilon_0}\right) > 0$$

So F_z is positive & $\vec{F}$ points up.

Since $\vec{F}$ points in the opposite direction to the displacement, then equilibrium is stable in this direction.